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STABILITY OF THE ONE-DIMENSIONAL MOTIONS OF A VISCOUS GAS WITH A LINEAR DEPENDENCE OF THE VELOCITY ON THE COORDINATES*

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The stability of a new solution of the equations of one-dimensional gas dynamics is investigated. This solution is a generalization of the solutions of Sedov /1, 2/ to the case of a viscous, thermally conducting ideal gas with an exponential dependence of the coefficient of viscosity and thermal conductivity on temperature. The linearized equations for small perturbations (the effects of thermal conductivity are not allowed for in the equations for the perturbations), which contain functions of time and the radial coordinate in the coefficients, can be solved by separation of the variables. The conditions under which instability arises are determined from an analysis of the time parts of the solutions.

The stability of the solutions /1/ has been considered in /3-5/.

1. We shall investigate the stability of the motions of a viscous, thermally conducting gas with a velocity distribution of the form

$$v_i^\circ = \frac{dR}{dt}\frac{x_i}{R} \tag{1.1}$$

(x_i are the Cartesian coordinates and $R(t)$ is a scale factor).

After the change of variables $(x_i, t) \to (y_i, t)$, where $y_i = x_i/R(t)$ are Lagrangian variables, the equations of motion are represented in the form

$$\frac{\partial\rho}{\partial t} + \frac{1}{R}\left(v_k - \frac{dR}{dt}y_k\right)\frac{\partial\rho}{\partial y_k} + \frac{\rho}{R}\frac{\partial v_k}{\partial y_k} = 0 \tag{1.2}$$

$$\rho\left[\frac{\partial v_i}{\partial t} + \frac{1}{R}\left(v_k - \frac{dR}{dt}y_k\right)\frac{\partial v_i}{\partial y_k}\right] = -\frac{1}{R}\frac{\partial p}{\partial y_i} + \frac{1}{R^2}\left\{\frac{\partial}{\partial y_i}\left[\left(\lambda - \frac{2}{3}\mu\right)\frac{\partial v_k}{\partial y_k}\right] + \frac{\partial}{\partial y_k}(2\mu e_{ik})\right\}$$

$$\rho\left[\frac{\partial E}{\partial t} + \frac{1}{R}\left(v_k - \frac{dR}{dt}y_k\right)\frac{\partial E}{\partial y_k}\right] = \frac{\gamma}{R^2}\frac{\partial}{\partial y_k}\left(\chi\frac{\partial E}{\partial y_k}\right) - \frac{p}{R}\frac{\partial v_k}{\partial y_k} + \frac{1}{R^2}\left[\left(\lambda - \frac{2}{3}\mu\right)\left(\frac{\partial v_k}{\partial y_k}\right)^2 + 2\mu e_{ik}e_{ik}\right]$$

$$p = (\gamma - 1)\rho E, \quad e_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial y_k} + \frac{\partial v_k}{\partial y_i}\right)$$

$$\lambda = a_1E^n, \quad \mu = a_2E^n, \quad \chi = a_3E^n$$

Here, λ is the coefficient of bulk viscosity, μ is the coefficient of shear strength, $\chi = \varkappa/c_p$, where $\varkappa$ is the thermal conductivity and E is the internal energy per unit mass. The remaining symbols have their usual meanings. The indices i and k cover the values $1, \ldots, \nu$, where the value of ν indicates the form of the symmetry of the problem ($\nu = 1, 2, 3$).

The solutions of Eq.(1.2) for motions with the velocity distribution (1.1) has the form

$$\rho^\circ = CR^{-\nu}\xi^{2n-2}, \quad E^\circ = U(t)\,\xi^2, \quad \xi = (y_ky_k)^{1/2} \tag{1.3}$$

$$\frac{dV}{dt} = 2nS, \quad \frac{dU}{dt} = \nu VS + \frac{2\gamma a_3}{C}(2n+\nu)R^{\nu-2}U^{n+1}$$

$$V = \frac{dR}{dt}, \; S = \frac{\nu A}{C}R^{\nu-2}U^nV - (\gamma - 1)\frac{U}{R}$$

$$A = a_1 - \frac{2a_2}{3} + \frac{2a_2}{\nu} \tag{1.4}$$

It follows from the equations for V and U that, when the gas is compressed $(V < 0)$, the magnitudes of $|V|$ and U increase monotonically while, when the gas expands $(V > 0)$, these quantities, depending on the initial conditions, may both increase as well as decrease and,

*Prikl.Matem.Mekhan., 51, 6, 951-956, 1987 in the latter case, there is a change in the sign of V.

It should be noted that for certain isolated values of the parameters in system (1.2) (when $n=0, n=1$), for example), there may be solutions which differ from (1.3). In this paper only solutions of (1.3) which are valid for any values of the parameters are investigated.

In order to investigate the stability, let us impose small perturbations

$$v_i = \frac{dR}{dt} y_i + u_i, \quad \rho = \rho^\circ (1+\Theta), \quad E = E^\circ (1+\varepsilon)$$

on the solution of (1.1), (1.3).

By neglecting the effects of thermal conduction $(a_3 = 0)$ in the equations for the perturbations and in the equation for U (1.3), we obtain from (1.2), after linearization with respect to the perturbations, convolution of the equations of motion with the operators y_i and $\partial/\partial y_i$ and a number of transformations, the following system of equations:

$$R^2 \frac{\partial \Theta}{\partial t} = -(2n-2)H - G \tag{1.5}$$

$$\begin{aligned}
R^2 \frac{\partial \varepsilon}{\partial t} &= -2H - (\gamma - 1)G + \\
&\frac{A\nu}{C} V U^{n-1} R^{\nu-1} [\nu V R (n\varepsilon - \Theta - \varepsilon) + 2G] \\
\frac{\partial H}{\partial t} &= -(\gamma-1)U\left(\frac{\partial \Theta}{\partial \eta} + \frac{\partial \varepsilon}{\partial \eta} + 2n\varepsilon\right) + \\
&\frac{1}{C} R^{\nu-2} U^n \left\{ a_1 \left(\frac{\partial G}{\partial \eta} + 2nG\right) + a_2 \left[\frac{1}{3}\frac{\partial G}{\partial \eta} - \right.\right. \\
&\left(\frac{4}{3} n + 2\right) G + \Delta_\eta H + (4n+4)\frac{\partial H}{\partial \eta} + (4n + 2\nu) H \Big] + \\
&\left. A\left[n\nu V R\left(\frac{\partial \varepsilon}{\partial \eta} + 2n\varepsilon - 2\Theta\right)\right]\right\} \\
\frac{\partial G}{\partial t} &= -(\gamma-1)U\left[\Delta_\eta \Theta + 2\frac{\partial \Theta}{\partial \eta} + \Delta_\eta \varepsilon + (2n+2)\frac{\partial \varepsilon}{\partial \eta} + 2n\nu\varepsilon\right] + \\
&\frac{1}{C} R^{\nu-2} U^n \left\{ a_1 \left[\Delta_\eta G + 2(n+1)\frac{\partial G}{\partial \eta}\right] + \right. \\
&a_2 \left[\frac{4}{3}\Delta_\eta G + \frac{2}{3}(n+1)\frac{\partial G}{\partial \eta} - 4(n+1)G + \right. \\
&\left. 2(n+1)\left(\Delta_\eta H + 4\frac{\partial H}{\partial \eta} + 2\nu H\right)\right] + A \cdot 2n\nu G + \\
&\left. A n \nu V R \left[\Delta_\eta \varepsilon + 2(n+1)\frac{\partial \varepsilon}{\partial \eta} + 2n\nu\varepsilon - 2\frac{\partial \Theta}{\partial \eta} - 2\nu\Theta\right]\right\}
\end{aligned} \tag{1.6}$$

$$H = \frac{u_k y_k}{\xi^2} R, \quad G = \frac{\partial u_k}{\partial y_k} R, \quad \eta = \ln \xi \tag{1.7}$$

$$\Delta_\eta = \frac{\partial^2}{\partial \eta^2} + (\nu - 2)\frac{\partial}{\partial \eta} + \Delta_1$$

Here, Δ_1 denotes the angular part of the Laplacian operator.

2. Let us initially consider the solutions of system (1.5) and (1.6) for the case of unidimensional perturbations: $u = u_r(\xi, t)$, $\Theta = \Theta(\xi, t)$, $\varepsilon = \varepsilon(\xi, t)$. As a consequence of the relationships $G = \partial H/\partial \eta + \nu H$ and $\Delta_1 = 0$, both the equations of (1.6) turn out to be equivalent. System (1.5), (1.6) reduces to a single equation in the quantity H which enables one to carry out a separation of the variables

$$H = \sum_k H^{(k)}(t) f^{(k)}(\xi); \quad \left(\frac{\partial}{\partial \eta} + 2n\right)\left(\frac{\partial}{\partial \eta} + \nu\right) f^{(k)} = -k^2 f^{(k)} \tag{2.1}$$

$$f^{(k)} = \xi^{-(n+\nu/2)} \sin(b^{(k)} \ln \xi + \delta^{(k)}), \quad b^{(k)} = \left[k^2 - \left(n - \frac{\nu}{2}\right)^2\right]^{1/2}$$

The constants $b^{(k)}$ and $\delta^{(k)}$ are determined from the boundary conditions. For example, in the case of a gas bounded by solid surfaces at $r = R(t)$ and $r = \xi_* R(t)$, we obtain from the condition that $u_r = 0$ when $\xi = 1$ and $\xi = \xi_*$

$$\delta^{(k)} = 0, \quad b^{(k)} = \frac{\pi m}{\ln \xi_*}; \quad k^2 = \left(n - \frac{\nu}{2}\right)^2 + \left(\frac{\pi m}{\ln \xi_*}\right)^2$$

With the aim of simplifying the calculations, we shall consider the equations, defining $H^{(k)}(t)$ for the special case of a monatomic gas and spherical symmetry $(\gamma = 5/3,\ a_1 = 0,\ \nu = 3)$. It then follows from (1.4) that $A = 0$ and the expansions for U and V and the equations for $H^{(k)}$ are respresented in the form (Q_1 and Q_2 are constants)

$$U = Q_1 R^{-2}, \quad V = \pm\left(\frac{4}{3} n Q_1 R^{-2} + Q_2\right)^{1/2}$$

$$R^2 \frac{d}{dt} R^2 \frac{dH^{(k)}}{dt} + \frac{a_2 Q_1^n}{C}\left(\frac{4}{3}k^2 + 8n\right) R^2 \frac{d}{dt}\left(R^{3-2n} H^{(k)}\right) +$$
$$Q_1\left(\frac{10}{9}k^2 + \frac{4}{3}n\right) H^{(k)} = 0$$

Next, by transforming this equation, we shall, to be specific, consider perturbations which develop in the main motion at the stage when the gas is compressed $(V < 0)$. There are no difficulties in passing to the case of expansion in the final formulae.

By using the new independent variable τ, which increases monotonically with time under any compression regime, we get

$$\frac{d^2H^{(k)}}{d\tau^2} + Bg(\tau)\frac{dH^{(k)}}{d\tau} + [\omega^2 + Bg'(\tau)]\, H^{(k)} = 0 \tag{2.2}$$

$$\omega^2 = \frac{5k^2}{6n} + 1, \quad B = \begin{cases} a_2 Q_1 \,|\, Q_2 \,|^{n-3/2}/C, & Q_2 \neq 0 \\ a_2 Q_1^{n-1/2}/(C R_0^{2n-3}), & Q_2 = 0 \end{cases}$$

$$g(\tau) = \left(\frac{4}{3}n\right)^{2-n}\left(\frac{k^2}{n} + 6\right) b^{2n-3}(\tau)$$

$$Q_2 > 0, \quad \tau = \operatorname{arch}\left(\sqrt{4/_3 n Q_1/Q_2}\, R^{-1}\right), \quad b(\tau) = \operatorname{sh}\tau$$
$$Q_2 < 0, \quad \tau = \operatorname{arch}\left(\sqrt{4/_3 n Q_1/|\, Q_2\,|}\, R^{-1}\right), \quad b(\tau) = \operatorname{ch}\tau$$
$$Q_2 = 0, \quad \tau = \ln\left(\sqrt{4/_3 n}\, R_0 R^{-1}\right), \quad b(\tau) = e^\tau$$

(R_0 is a constant which is equal to the value of R at the instant of time $t = 0$).

In the case when $n = 0$, Eq. (2.2) has the same form while the remaining formulae are replaced by the following:

$$\omega^2 = {}^5/_6 k^2, \quad B = a_2 Q_1 Q_2^{-3/2}/C$$
$$g(\tau) = ({}^4/_3)^2 k^2 b^{-3}(\tau), \quad \tau = \sqrt{4/_3 Q_1/Q_2}\, R^{-1}, \; b(\tau) = \tau$$

Let us investigate the nature of the solutions of Eq. (2.2). In the case of adiabatic motions $(B = 0)$, the solution has the form $H^{(k)} = \sin(\omega\tau + \varphi_0)$. Since, the radial component of the perturbation of the velocity is associated with the quantity H by the relationship $u_r = \xi H R^{-1}$ (see (1.7)), we get

$$u^{(k)} = R^{-1} H^{(k)} = M \sin\varphi; \quad M = M_0\, b(\tau), \; \varphi = \omega\tau + \varphi_0$$

for the time part of the perturbation of the velocity $u^{(k)}$.

It follows from this expression that, during an adiabatic compression of the gas, the perturbation of the velocity increases in an oscillatory manner with time.

The nature of the solutions of Eq. (2.2) when $B \neq 0$ can be represented by considering perturbations with large k and small values of the parameter $B \sim 1/k^2$ so that $\omega \gg 1, Bg \sim 1$. By applying the method of averaging /6/ up to the second approximation, we obtain that the amplitude and phase the fundamental harmonic are determined by the expressions

$$M = M_0 b(\tau) \exp\left(-\frac{B}{2}\int_{\tau_0}^{\tau} g(\tau)\, d\tau\right) \tag{2.3}$$
$$\varphi = \omega\left\{\tau + \frac{1}{\omega^2}\left[\frac{B}{2} g(\tau) - \frac{B^2}{8}\int_{\tau_0}^{\tau} g^2(\tau)\, d\tau\right]\right\} + \varphi_0$$

The solution of (2.3) corresponds to vibrations which increase subject to the condition that $dM/dt > 0$. This relationship, taking into account the definition of τ in (2.2) is represented in the form

$$\left(\frac{4}{3} n Q_1 + Q_2 R^2\right)^{1/2} - \frac{2}{3} k^2 \frac{a_2 Q_1^n}{C} R^{-(2n-3)} > 0 \tag{2.4}$$

It can be seen from this that, at the initial instant of time, the possibility that the perturbations will grow is determined by the relationship between the wave number of the perturbation k, the viscosity of the gas a_2 and the constants C, Q_1, Q_2 and R_0 which specify the values of the density, energy and the compression regime. The change in the stability properties on further compression depends, in the first place, on the magnitude of the index n in the law governing the change in viscosity with temperature.

When $n \geqslant 3/_2$, the motion, which is stable at the initial instant of time (when $t = 0$, the left-hand side of (2.4) is negative for any k), also subsequently remains stable.

When $n < {}^3/_2$, compression of the gas leads to the occurrence of instability depending on the initial conditions since the left-hand side of (2.4) becomes positive for any k.

In considering the behaviour of the perturbations at stages corresponding to an expansion of the gas in the main motion $(V > 0)$, it is necessary to take account of the fact that, in this case, on passing to Eq. (2.2), $R^2 d/dt = -\sqrt{{}^4/_3 Q_1}\, d/d\tau$, i.e. the sign in front of B changes. Consequently, by changing the sign in front of B in (2.3) and allowing for the fact that, on passing to (2.4), $d\tau/dt < 0$, it is possible to draw a conclusion concerning the attenuation of the perturbations of the velocity on the background of the expansion of the gas for any k and any values of the parameters for the problem.

It should be noted that the conclusions which has been drawn remain true while the change in the perturbations with time occurs, as in the adiabatic case, in an oscillatory manner. The negative component, which occurs in the expression for the frequency $d\varphi/dt$, indicates that, at the initial instant of time, the dependence of the perturbations on time is only of an oscillatory nature for not large values of B. When $n < {}^3/_2$, the magnitude of this component decreases with time, i.e. the dependence $u^{(k)}(t)$ is oscillatory in this case. When $n > {}^3/_2$, this component increases, i.e. at high degrees of compression, the dependence of the perturbations on time will be monotonic. When $n = {}^3/_2$, the magnitude of the component is constant.

The validity of the conclusions drawn on the basis of the approximate expressions (2.3) can be confirmed by the exact solutions of the equations for $H^{(k)}$ which exist for individual values of the parameters.

When $n = {}^3/_2$, (2.2) is converted into an equation with constant coefficients and its solution yields an expression for the amplitude of the vibrations which is identical to (2.3). After expanding the expression for the frequency in series in $1/\omega^2$, it is also identical to the quantity $d\varphi/dt$, determined from (2.3). It is obvious from the same expression that the oscillatory nature of the time dependence is conserved if $Bg < 2\omega$.

When $Bg > 2\omega$, by calculating the quantity $u^{(k)} = R^{-1}H^{(k)}$ for sufficiently large τ when $R^{-1} \sim e^{\tau}$ (see (2.2)), it can be seen that the perturbations of the velocity increase monotonically subject to the condition $1 - Bg/2 + \sqrt{(Bg)^2/4 - \omega^2} > 0$ or $Bg > \omega^2 + 1$. Hence, when $n = {}^3/_2$, an increase in viscosity is a stabilizing factor in the domain $Bg < 2\omega$ (oscillatory modes) and a destabilizing factor in the domain $Bg > 2\omega$, when the oscillatory modes are converted into monotonic modes.

When $Q_2 = 0$, (2.2) is transformed into a degenerate hypergeometric equation, the solution of which has the form

$$w = c_1\Phi(\sigma, 2\sigma + 1, z) + c_2\Psi(\sigma, 2\sigma + 1, z), \quad n > {}^3/_2$$
$$w = e^z[c_1\Phi(\sigma + 1, 2\sigma + 1, -z) + c_2\Psi(\sigma + 1, 2\sigma + 1, -z)],$$
$$n < {}^3/_2$$
$$w(z) = z^{-\sigma}e^z H^{(k)}, \quad z = \frac{Bg(\tau)}{2n-3}, \quad \sigma = i\frac{\omega}{2n-3}$$

We shall use the asymptotic representations for Φ and Ψ /8/ which correspond to large degrees of compression: $z \to \infty\ (n > {}^3/_2)$, $\Phi(\alpha, \beta, z) \sim e^z z^{\alpha-\beta}$, $\Psi(\alpha, \beta, z) \sim z^{-\alpha}$; $z \to 0\ (n < {}^3/_2)$, $\Phi(\alpha, \beta, z) \to 1$, $\Psi(\alpha, \beta, z)\, z^{1-\beta}$.

Taking account of the definition of the variables w and z and the relationships $u^{(k)} = R^{-1}H^{(k)}$, we obtain, in the case of the time part of the perturbation of the velocity

$$u^{(k)} = c_1'\zeta^{4-2n} + c_2'\zeta \exp\left[-B\frac{\sqrt{{}^4/_3 n}}{2n-3}\left(\frac{k^2}{n} + 6\right)\zeta^{2n-3}\right], \quad n > \frac{3}{2}$$
$$u^{(k)} = M_0\zeta \sin(\omega \ln \zeta + \varphi_0), \quad n < \frac{3}{2}; \quad \zeta = \frac{R_0}{R}$$

(c_1' and c_2' are constants).

It can be seen from the first expression that, when $n > {}^3/_2$, the oscillatory nature of the time dependence of the perturbations is replaced by a monotonic dependence when there is sufficient compression of the gas. It follows from the second expression that, when $n < {}^3/_2$, compression leads to an oscillatory increase in the perturbations of the velocity for any k for any values of the parameters. Hence, the validity of the conclusions which were drawn on the basis of the approximate expressions (2.3) is not confined to the domain of large k and small B.

The analysis of the time parts of the perturbations which has been presented above referred to the case of a monatomic gas and spherical symmetry. In other cases, the threshold value of n differs from ${}^3/_2$.

Let us now briefly consider the changes in the nature of the dependence $u^{(k)}(t)$ which are due to the non-univariate nature of the perturbations. Restricting ourselves to the case of a monatomic gas and spherical symmetry and, for simplicity, only considering the value $n = {}^3/_2$, we reduce system (1.5), (1.6) to a single equation in H. This equation can be solved by separation of the variables

$$H = \sum_k H^{(k)}(t) f^{(k)}(\xi) Y_l(\vartheta, \varphi)$$

where $Y_l(\vartheta, \varphi)$ are spherical harmonics and the functions $f^{(k)}(\xi)$ are defined in (2.1).

The equations for $H^{(k)}$ have the form

$$\frac{d^4H^{(k)}}{d\tau^4} + \frac{B}{\sqrt{2}}(q_1 + q_2)\frac{d^3H^{(k)}}{d\tau^3} + \left\{\frac{q_3}{3} + \frac{B^2}{2}\left[q_1q_2 - \frac{70}{3}l(l+1)\right]\right\}\frac{d^2H^{(k)}}{d\tau^2} + \frac{B}{3\sqrt{2}}\left[q_3q_2 - \frac{62}{3}l(l+1)\right]\frac{dH^{(k)}}{d\tau} - \frac{4}{9}l(l+1)H^{(k)} = 0$$

$$q_1 = {}^4/_3h^2 + 12, \quad q_2 = h^2 + 7, \quad q_3 = {}^5/_3h^2 + 3; \quad h^2 = k^2 + l(l+1)$$

where τ and B are defined in (2.2).

Above all, it is obvious from an analysis of the characteristic equation that a pair of real roots appears in the case of non-univariate perturbations which, for sufficiently small B, correspond to monotonic modes of instability.

The effect of the non-univariate character of the perturbations on the vibrational modes involves a change in the frequency of the vibrations and the size of the decrement in the factor describing the viscous damping. In the case of the most critical lower levels of the spectrum of perturbations (small k), smaller values of the decrement correspond to univariate perturbations.

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